A 4-cycle in the absence of 3-cycles would require nonadjacent vertices with two common neighbors, which Proposition 1.1.38 forbids. Finally, the vertices $12, 34, 51, 23, 45$ yield a 5-cycle, so the girth is 5. ∎

The Petersen graph is highly symmetric. Every permutation of $\{1, 2, 3, 4, 5\}$ generates a permutation of the 2-subsets that preserves the disjointness relation. Thus there are at least $5! = 120$ isomorphisms from the Petersen graph to itself. Exercise 43 confirms that there are no others.

1.1.41.* Definition. An **automorphism** of G is an isomorphism from G to G. A graph G is **vertex-transitive** if for every pair $u, v \in V(G)$ there is an automorphism that maps u to v.

The automorphisms of G are the permutations of $V(G)$ that can be applied to both the rows and the columns of $A(G)$ without changing $A(G)$.

1.1.42.* Example. *Automorphisms.* Let G be the path with vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 34\}$. This graph has two automorphisms: the identity permutation and the permutation that switches 1 with 4 and switches 2 with 3. Interchanging vertices 1 and 2 is not an automorphism of G, although G is isomorphic to the graph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{21, 13, 34\}$.

In $K_{r,s}$, permuting the vertices of one partite set does not change the adjacency matrix; this leads to $r!s!$ automorphisms. When $r = s$, we can also interchange the partite sets; $K_{t,t}$ has $2(t!)^2$ automorphisms.

The biclique $K_{r,s}$ is vertex-transitive if and only if $r = s$. If $n > 2$, then P_n is not vertex-transitive, but every cycle is vertex-transitive. The Petersen graph is vertex-transitive. ∎

We can prove a statement for every vertex in a vertex-transitive graph by proving it for one vertex. Vertex-transitivity guarantees that the graph "looks the same" from each vertex.

EXERCISES

Solutions to problems generally require clear explanations written in sentences. The designations on problems have the following meanings:

"(−)" = easier or shorter than most,
"(+)" = harder or longer than most,
"(!)" = particularly useful or instructive,
"(∗)" = involves concepts marked optional in the text.

The exercise sections begin with easier problems to check understanding, ending with a line of dots. The remaining problems roughly follow the order of material in the text.

1.1.1. (−) Determine which complete bipartite graphs are complete graphs.

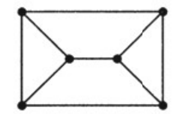
1.1.2. (−) Write down all possible adjacency matrices and incidence matrices for a 3-vertex path. Also write down an adjacency matrix for a path with six vertices and for a cycle with six vertices.

1.1.3. (−) Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

1.1.4. (−) From the definition of isomorphism, prove that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

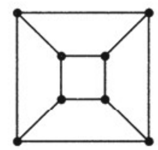
1.1.5. (−) Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle.

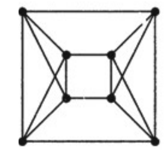
1.1.6. (−) Determine whether the graph below decomposes into copies of P_4.



1.1.7. (−) Prove that a graph with more than six vertices of odd degree cannot be decomposed into three paths.

1.1.8. (−) Prove that the 8-vertex graph on the left below decomposes into copies of $K_{1,3}$ and also into copies of P_4.





1.1.9. (−) Prove that the graph on the right above is isomorphic to the complement of the graph on the left.

1.1.10. (−) Prove or disprove: The complement of a simple disconnected graph must be connected.

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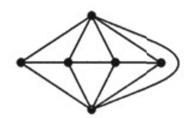
1.1.11. Determine the maximum size of a clique and the maximum size of an independent set in the graph below.



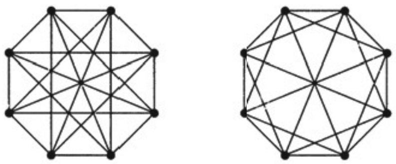
1.1.12. Determine whether the Petersen graph is bipartite, and find the size of its largest independent set.

1.1.13. Let G be the graph whose vertex set is the set of k-tuples with coordinates in $\{0, 1\}$, with x adjacent to y when x and y differ in exactly one position. Determine whether G is bipartite.

1.1.14. (!) Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be partitioned into 1-by-2 and 2-by-1 rectangles. Using the same argument, make a general statement about all bipartite graphs.

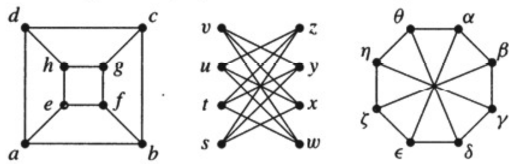
1.1.15. Consider the following four families of graphs: $A = \{\text{paths}\}$, $B = \{\text{cycles}\}$, $C = \{\text{complete graphs}\}$, $D = \{\text{bipartite graphs}\}$. For each pair of these families, determine all isomorphism classes of graphs that belong to both families.

1.1.16. Determine whether the graphs below are isomorphic.



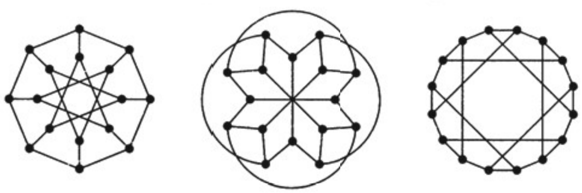
1.1.17. Determine the number of isomorphism classes of simple 7-vertex graphs in which every vertex has degree 4.

1.1.18. Determine which pairs of graphs below are isomorphic.

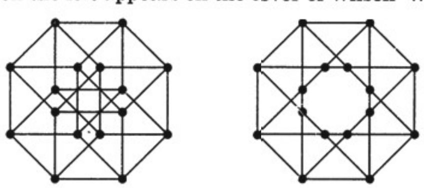


1.1.19. Determine which pairs of graphs below are isomorphic.

1.1.20. Determine which pairs of graphs below are isomorphic.



1.1.21. Determine whether the graphs below are bipartite and whether they are isomorphic. (The graph on the left appears on the cover of Wilson–Watkins [1990].)



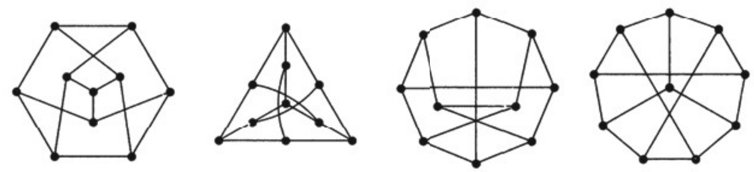
1.1.22. (!) Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest possible number of pairs.

1.1.23. In each class below, determine the smallest n such that there exist nonisomorphic n-vertex graphs having the same list of vertex degrees.

(a) all graphs, (b) loopless graphs, (c) simple graphs.

(Hint: Since each class contains the next, the answers form a nondecreasing triple. For part (c), use the list of isomorphism classes in Example 1.1.31.)

1.1.24. (!) Prove that the graphs below are all drawings of the Petersen graph (Definition 1.1.36). (Hint: Use the disjointness definition of adjacency.)



1.1.25. (!) Prove that the Petersen graph has no cycle of length 7.

1.1.26. (!) Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least $2k$ vertices. Determine all such graphs with exactly $2k$ vertices.

1.1.27. (!) Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with exactly $k^2 + 1$ vertices.

1.1.28. (+) *The Odd Graph* O_k. The vertices of the graph O_k are the k-element subsets of $\{1, 2, \ldots, 2k + 1\}$. Two vertices are adjacent if and only if they are disjoint sets. Thus O_2 is the Petersen graph. Prove that the girth of O_k is 6 if $k \geq 3$.

1.1.29. Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

1.1.30. Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry in A^2 and in MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

1.1.31. (!) Prove that a self-complementary graph with n vertices exists if and only if n or $n - 1$ is divisible by 4. (Hint: When n is divisible by 4, generalize the structure of P_4 by splitting the vertices into four groups. For $n \equiv 1 \bmod 4$, add one vertex to the graph constructed for $n - 1$.)

1.1.32. Determine which bicliques decompose into two isomorphic subgraphs.

1.1.33. For $n = 5$, $n = 7$, and $n = 9$, decompose K_n into copies of C_n.

1.1.34. (!) Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4.

1.1.35. (!) Prove that K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3. (Hint: For the case where n is divisible by 3, split the vertices into three sets of equal size.)

1.1.36. Prove that if K_n decomposes into triangles, then $n-1$ or $n-3$ is divisible by 6.

1.1.37. Let G be a graph in which every vertex has degree 3. Prove that G has no decomposition into paths that each have at least 5 vertices.

1.1.38. (!) Let G be a simple graph in which every vertex has degree 3. Prove that G decomposes into claws if and only if G is bipartite.

1.1.39. (+) Determine which of the graphs in Example 1.1.35 can be used to form a decomposition of K_6 into pairwise-isomorphic subgraphs. (Hint: Each graph that is not excluded by some divisibility condition works.)

1.1.40. (∗) Count the automorphisms of P_n, C_n, and K_n.

1.1.41. (∗) Construct a simple graph with six vertices that has only one automorphism. Construct a simple graph that has exactly three automorphisms. (Hint: Think of a rotating triangle with appendages to prevent flips.)

1.1.42. (∗) Verify that the set of automorphisms of G has the following properties:
a) The composition of two automorphisms is an automorphism.
b) The identity permutation is an automorphism.
c) The inverse of an automorphism is also an automorphism.
d) Composition of automorphisms satisfies the associative property.
(Comment: Thus the set of automorphisms satisfies the defining properties for a *group*.)

1.1.43. (∗) *Automorphisms of the Petersen graph.* Consider the Petersen graph as defined by disjointness of 2-sets in $\{1,2,3,4,5\}$. Prove that every automorphism maps the 5-cycle with vertices $12, 34, 51, 23, 45$ to a 5-cycle with vertices ab, cd, ea, bc, de determined by a permutation of $\{1,2,3,4,5\}$ taking elements 1,2,3,4,5 to a, b, c, d, e, respectively. (Comment: This implies that there are only 120 automorphisms.)

1.1.44. (∗) The Petersen graph has even more symmetry than vertex-transitivity. Let $P = (u_0, u_1, u_2, u_3)$ and $Q = (v_0, v_1, v_2, v_3)$ be paths with three edges in the Petersen graph. Prove that there is exactly one automorphism of the Petersen graph that maps u_i into v_i for $i = 0, 1, 2, 3$. (Hint: Use the disjointness description.)

1.1.45. (∗) Construct a graph with 12 vertices in which every vertex has degree 3 and the only automorphism is the identity.

1.1.46. (∗) *Edge-transitivity.* A graph G is **edge-transitive** if for all $e, f \in E(G)$ there is an automorphism of G that maps the endpoints of e to the endpoints of f (in either order). Prove that the graphs of Exercise 1.1.21 are vertex-transitive and edge-transitive. (Comment: Complete graphs, bicliques, and the Petersen graph are edge-transitive.)

1.1.47. (∗) *Edge-transitive versus vertex-transitive.*
a) Let G be obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2. Prove that G is edge-transitive but not vertex-transitive.
b) Suppose that G is edge-transitive but not vertex-transitive and has no vertices of degree 0. Prove that G is bipartite.
c) Prove that the graph in Exercise 1.1.6 is vertex-transitive but not edge-transitive.

1.2. Paths, Cycles, and Trails

In this section we return to the Königsberg Bridge Problem, determining when it is possible to traverse all the edges of a graph. We also we develop useful properties of connection, paths, and cycles.

Before embarking on this, we review an important technique of proof. Many statements in graph theory can be proved using the principle of induction. Readers unfamiliar with induction should read the material on this proof technique in Appendix A. Here we describe the form of induction that we will use most frequently, in order to familiarize the reader with a template for proof.

1.2.1. Theorem. (Strong Principle of Induction). Let $P(n)$ be a statement with an integer parameter n. If the following two conditions hold, then $P(n)$ is true for each positive integer n.
1) $P(1)$ is true.
2) For all $n > 1$, "$P(k)$ is true for $1 \leq k < n$" implies "$P(n)$ is true".

Proof: We ASSUME the **Well Ordering Property** for the positive integers: every nonempty set of positive integers has a least element. Given this, suppose that $P(n)$ fails for some n. By the Well Ordering Property, there is a least n such that $P(n)$ fails. Statement (1) ensures that this value cannot be 1. Statement (2) ensures that this value cannot be greater than 1. The contradiction implies that $P(n)$ holds for every positive integer n. ∎

In order to apply induction, we verify (1) and (2) for our sequence of statements. Verifying (1) is the **basis step** of the proof; verifying (2) is the **induction step**. The statement "$P(k)$ is true for all $k < n$" is the **induction hypothesis**, because it is the hypothesis of the implication proved in the induction step. The variable that indexes the sequence of statements is the **induction parameter**.

The induction parameter may be any integer function of the instances of our problem, such as the number of vertices or edges in a graph. We say that we are using "induction on n" when the induction parameter is n.

There are many ways to phrase inductive proofs. We can start at 0 to prove a statement for nonnegative integers. When our proof of $P(n)$ in the induction step makes use only of $P(n-1)$ from the induction hypothesis, the technique is called "ordinary" induction; making use of all previous statements is "strong" induction. We seldom distinguish between strong induction and ordinary induction; they are equivalent (see Appendix A).

Most students first learn ordinary induction in the following phrasing: 1) verify that $P(n)$ is true when $n = 1$, and 2) prove that if $P(n)$ is true when n is k, then $P(n)$ is also true when n is $k+1$. Proving $P(k+1)$ from $P(k)$ for $k \geq 1$ is equivalent to proving $P(n)$ from $P(n-1)$ for $n > 1$.

Since T is closed, there is a trail T' that starts and ends at v and uses the same edges as T. We now extend T' along e' to obtain a longer trail than T. This contradicts the choice of T, and hence T traverses all edges of G. ■

This proof and the resulting construction procedure (Exercise 12) are similar to those of Hierholzer [1873]. Exercise 35 develops another proof.

Later chapters contain several applications of the statement that every connected even graph has an Eulerian circuit. Here we give a simple one. When drawing a figure G on paper, how many times must we stop and move the pen? We are not allowed to repeat segments of the drawing, so each visit to the paper contributes a trail. Thus we seek a decomposition of G into the minimum number of trails. We may reduce the problem to connected graphs, since the number of trails needed to draw G is the sum of the number needed to draw each component.

For example, the graph G below has four odd vertices and decomposes into two trails. Adding the dashed edges on the right makes it Eulerian.

G  G'

1.2.33. Theorem. For a connected nontrivial graph with exactly $2k$ odd vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$.

Proof: A trail contributes even degree to every vertex, except that a non-closed trail contributes odd degree to its endpoints. Therefore, a partition of the edges into trails must have some non-closed trail ending at each odd vertex. Since each trail has only two ends, we must use at least k trails to satisfy $2k$ odd vertices. We also need at least one trail since G has an edge, and Theorem 1.2.26 implies that one trail suffices when $k = 0$.

It remains to prove that k trails suffice when $k > 0$. Given such a graph G, we pair up the odd vertices in G (in any way) and form G' by adding for each pair an edge joining its two vertices, as illustrated above. The resulting graph G' is connected and even, so by Theorem 1.2.26 it has an Eulerian circuit C. As we traverse C in G', we start a new trail in G each time we traverse an edge of $G' - E(G)$. This yields k trails decomposing G. ■

We prove theorems in general contexts to avoid work. The proof of Theorem 1.2.33 illustrates this; by transforming G into a graph where Theorem 1.2.26 applies, we avoid repeating the basic argument of Theorem 1.2.26. Exercise 33 requests a proof of Theorem 1.2.33 directly by induction.

Note that Theorem 1.2.33 considers only graphs having an even number of vertices of odd degree. Our first result in the next section explains why.

EXERCISES

Most problems in this book require proofs. Words like "construct", "show", "obtain", "determine", etc., explicitly state that proof is required. Disproof by providing a counterexample requires confirming that it is a counterexample.

1.2.1. (−) Determine whether the statements below are true or false.

a) Every disconnected graph has an isolated vertex.

b) A graph is connected if and only if some vertex is connected to all other vertices.

c) The edge set of every closed trail can be partitioned into edge sets of cycles.

d) If a maximal trail in a graph is not closed, then its endpoints have odd degree.

1.2.2. (−) Determine whether K_4 contains the following (give an example or a proof of non-existence).

a) A walk that is not a trail.

b) A trail that is not closed and is not a path.

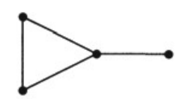
c) A closed trail that is not a cycle.

1.2.3. (−) Let G be the graph with vertex set $\{1, \ldots, 15\}$ in which i and j are adjacent if and only if their greatest common factor exceeds 1. Count the components of G and determine the maximum length of a path in G.

1.2.4. (−) Let G be a graph. For $v \in V(G)$ and $e \in E(G)$, describe the adjacency and incidence matrices of $G - v$ and $G - e$ in terms of the corresponding matrices for G.

1.2.5. (−) Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Conclude that no graph has a cut-vertex of degree 1.

1.2.6. (−) In the graph below (the paw), find all the maximal paths, maximal cliques, and maximal independent sets. Also find all the maximum paths, maximum cliques, and maximum independent sets.



1.2.7. (−) Prove that a bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.

1.2.8. (−) Determine the values of m and n such that $K_{m,n}$ is Eulerian.

1.2.9. (−) What is the minimum number of trails needed to decompose the Petersen graph? Is there a decomposition into this many trails using only paths?

1.2.10. (−) Prove or disprove:

a) Every Eulerian bipartite graph has an even number of edges.

b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

1.2.11. (−) Prove or disprove: If G is an Eulerian graph with edges e, f that share a vertex, then G has an Eulerian circuit in which e, f appear consecutively.

1.2.12. (−) Convert the proof at 1.2.32 to an procedure for finding an Eulerian circuit in a connected even graph.

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1.2.13. *Alternative proofs that every u,v-walk contains a u,v-path (Lemma 1.2.5).*

a) (ordinary induction) Given that every walk of length $l-1$ contains a path from its first vertex to its last, prove that every walk of length l also satisfies this.

b) (extremality) Given a u,v-walk W, consider a shortest u,v-walk contained in W.

1.2.14. Prove or disprove the following statements about simple graphs. (Comment: "Distinct" does not mean "disjoint".)

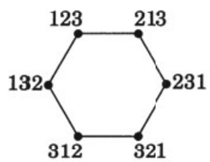
a) The union of the edge sets of distinct u,v-walks must contain a cycle.

b) The union of the edge sets of distinct u,v-paths must contain a cycle.

1.2.15. (!) Let W be a closed walk of length at least 1 that does not contain a cycle. Prove that some edge of W repeats immediately (once in each direction).

1.2.16. Let e be an edge appearing an odd number of times in a closed walk W. Prove that W contains the edges of a cycle through e.

1.2.17. (!) Let G_n be the graph whose vertices are the permutations of $\{1,\dots,n\}$, with two permutations $a_1,\dots,a_n$ and $b_1,\dots,b_n$ adjacent if they differ by interchanging a pair of adjacent entries (G_3 shown below). Prove that G_n is connected.



1.2.18. (!) Let G be the graph whose vertex set is the set of k-tuples with elements in $\{0,1\}$, with x adjacent to y if x and y differ in exactly two positions. Determine the number of components of G.

1.2.19. Let r and s be natural numbers. Let G be the simple graph with vertex set $v_0,\dots,v_{n-1}$ such that $v_i \leftrightarrow v_j$ if and only if $|j-i| \in \{r,s\}$. Prove that S has exactly k components, where k is the greatest common divisor of $\{n,r,s\}$.

1.2.20. (!) Let v be a cut-vertex of a simple graph G. Prove that $\overline{G}-v$ is connected.

1.2.21. Let G be a self-complementary graph. Prove that G has a cut-vertex if and only if G has a vertex of degree 1. (Akiyama–Harary [1981])

1.2.22. Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

1.2.23. For each statement below, determine whether it is true for every connected simple graph G that is not a complete graph.

a) Every vertex of G belongs to an induced subgraph isomorphic to P_3.

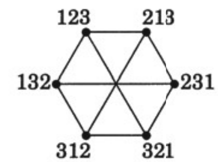
b) Every edge of G belongs to an induced subgraph isomorphic to P_3.

1.2.24. Let G be a simple graph having no isolated vertex and no induced subgraph with exactly two edges. Prove that G is a complete graph.

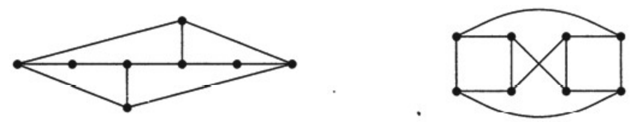
1.2.25. (!) Use ordinary induction on the number of edges to prove that absence of odd cycles is a sufficient condition for a graph to be bipartite.

1.2.26. (!) Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

1.2.27. Let G_n be the graph whose vertices are the permutations of $\{1,\dots,n\}$, with two permutations $a_1,\dots,a_n$ and $b_1,\dots,b_n$ adjacent if they differ by switching two entries. Prove that G_n is bipartite (G_3 shown below). (Hint: For each permutation a, count the pairs i,j such that $i<j$ and $a_i>a_j$; these are called **inversions**.)



1.2.28. (!) In each graph below, find a bipartite subgraph with the maximum number of edges. Prove that this is the maximum, and determine whether this is the only bipartite subgraph with this many edges.



1.2.29. (!) Let G be a connected simple graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique (complete bipartite graph).

1.2.30. Let G be a simple graph with vertices $v_1,\dots,v_n$. Let A^k denote the kth power of the adjacency matrix of G under matrix multiplication. Prove that entry i,j of A^k is the number of v_i,v_j-walks of length k in G. Prove that G is bipartite if and only if, for the odd integer r nearest to n, the diagonal entries of A^r are all 0. (Reminder: A walk is an **ordered** list of vertices and edges.)

1.2.31. (!) *Non-inductive proof of Theorem 1.2.23* (see Example 1.2.21).

a) Given $n \le 2^k$, encode the vertices of K_n as distinct binary k-tuples. Use this to construct k bipartite graphs whose union is K_n.

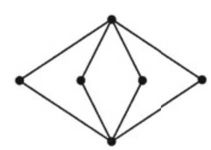
b) Given that K_n is a union of bipartite graphs $G_1,\dots,G_k$, encode the vertices of K_n as distinct binary k-tuples. Use this to prove that $n \le 2^k$.

1.2.32. The statement below is false. Add a hypothesis to correct it, and prove the corrected statement.

"Every maximal trail in an even graph is an Eulerian circuit."

1.2.33. Use ordinary induction on k or on the number of edges (one by one) to prove that a connected graph with $2k$ odd vertices decomposes into k trails if $k>0$. Does this remain true without the connectedness hypothesis?

1.2.34. Two Eulerian circuits are *equivalent* if they have the same unordered pairs of consecutive edges, viewed cyclically (the starting point and direction are unimportant). A cycle, for example, has only one equivalence class of Eulerian circuits. How many equivalence classes of Eulerian circuits are there in the graph drawn below?



1.2.35. *Tucker's Algorithm.* Let G be a connected even graph. At each vertex, partition the incident edges into pairs (each edge appears in a pair for each of its endpoints). Starting along a given edge e, form a trail by leaving each vertex along the edge paired with the edge just used to enter it, ending with the edge paired with e. This decomposes G into closed trails. As long as there is more than one trail in the decomposition, find two trails with a common vertex and combine them into a longer trail by changing the pairing at a common vertex. Prove that this procedure works and produces an Eulerian circuit as its final trail. (Tucker [1976])

1.2.36. (+) *Alternative characterization of Eulerian graphs.*

a) Prove that if G is Eulerian and $G' = G - uv$, then G' has an odd number of u,v-trails that visit v only at the end. Prove also that the number of the trails in this list that are not paths is even. (Toida [1973])

b) Let v be a vertex of odd degree in a graph. For each edge e incident to v, let $c(e)$ be the number of cycles containing e. Use $\sum_e c(e)$ to prove that $c(e)$ is even for some e incident to v. (McKee [1984])

c) Use part (a) and part (b) to conclude that a nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles.

1.2.37. (!) Use extremality to prove that the connection relation is transitive. (Hint: Given a u,v-path P and a v,w-path Q, consider the first vertex of P in Q.)

1.2.38. (!) Prove that every n-vertex graph with at least n edges contains a cycle.

1.2.39. Suppose that every vertex of a loopless graph G has degree at least 3. Prove that G has a cycle of even length. (Hint: Consider a maximal path.) (P. Kwok)

1.2.40. (!) Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

1.2.41. Let G be a connected graph with at least three vertices. Prove that G has two vertices x, y such that 1) $G - \{x, y\}$ is connected and 2) x, y are adjacent or have a common neighbor. (Hint: Consider a longest path.) (Chung [1978a])

1.2.42. Let G be a connected simple graph that does not have P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices. (Hint: Consider a vertex of maximum degree.) (Wolk [1965])

1.2.43. (+) Use induction on k to prove that every connected simple graph with an even number of edges decomposes into paths of length 2. Does the conclusion remain true if the hypothesis of connectedness is omitted?

1.3. Vertex Degrees and Counting

The degrees of the vertices are fundamental parameters of a graph. We repeat the definition in order to introduce important notation.

1.3.1. Definition. The **degree** of vertex v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v, except that each loop at v counts twice. The maximum degree is $\Delta(G)$, the minimum degree is $\delta(G)$, and G is **regular** if $\Delta(G) = \delta(G)$. It is k**-regular** if the common degree is k. The **neighborhood** of v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v.

1.3.2. Definition. The **order** of a graph G, written $n(G)$, is the number of vertices in G. An n**-vertex graph** is a graph of order n. The **size** of a graph G, written $e(G)$, is the number of edges in G. For $n \in \mathbb{N}$, the notation $[n]$ indicates the set $\{1, \ldots, n\}$.

Since our graphs are finite, $n(G)$ and $e(G)$ are well-defined nonnegative integers. We also often use "e" by itself to denote an edge. When e denotes a particular edge, it is not followed by the name of a graph in parentheses, so the context indicates the usage. We have used "n-cycle" to denote a cycle with n vertices; this is consistent with "n-vertex graph".

COUNTING AND BIJECTIONS

We begin with counting problems about subgraphs in a graph. The first such problem is to count the edges; we do this using the vertex degrees. The resulting formula is an essential tool of graph theory, sometimes called the "First Theorem of Graph Theory" or the "Handshaking Lemma".

1.3.3. Proposition. (Degree-Sum Formula) If G is a graph, then

$$\textstyle\sum_{v \in V(G)} d(v) = 2e(G).$$

Proof: Summing the degrees counts each edge twice, since each edge has two ends and contributes to the degree at each endpoint. ■

The proof holds even when G has loops, since a loop contributes 2 to the degree of its endpoint. For a loopless graph, the two sides of the formula count the set of pairs (v, e) such that v is an endpoint of e, grouped by vertices or grouped by edges. "Counting two ways" is an elegant technique for proving integer identities (see Exercise 31 and Appendix A).

The degree-sum formula has several immediate corollaries. Corollary 1.3.5 applies in Exercises 9–13 and in many arguments of later chapters.

1.3.4. Corollary. In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence $\delta(G) \le \frac{2e(G)}{n(G)} \le \Delta(G)$. ■

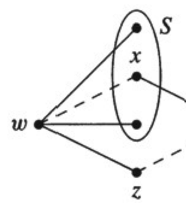
1.3.5. Corollary. Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree. ■

1.3.6. Corollary. A k-regular graph with n vertices has $nk/2$ edges. ■

We next introduce an important family of graphs.

1.3.7. Definition. The k**-dimensional cube** or **hypercube** Q_k is the simple graph whose vertices are the k-tuples with entries in $\{0, 1\}$ and whose

To find the modification when $N(w) \neq S$, we choose $x \in S$ and $z \notin S$ so that $w \leftrightarrow z$ and $w \not\leftrightarrow x$. We want to add wx and delete wz, but we must preserve vertex degrees. Since $d(x) \geq d(z)$ and already w is a neighbor of z but not x, there must be a vertex y adjacent to x but not to z. Now we delete $\{wz, xy\}$ and add $\{wx, yz\}$ to increase $|N(w) \cap S|$. ■

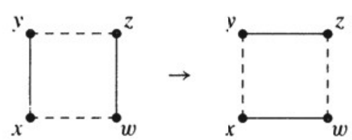


Theorem 1.3.31 tests a list of n numbers by testing a list of $n-1$ numbers; it yields a recursive algorithm to test whether d is graphic. The necessary condition "$\sum d_i$ even" holds implicitly: $\sum d_i' = (\sum d_i) - 2\Delta$ implies that $\sum d_i'$ and $\sum d_i$ have the same parity.

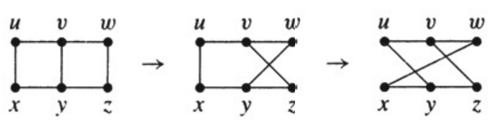
An algorithmic proof using "local change" pushes an object toward a desired condition. This can be phrased as proof by induction, where the induction parameter is the "distance" from the desired condition. In the proof of Theorem 1.3.31, this distance is the number of vertices in S that are missing from $N(w)$.

We used edge switches to transform an arbitrary graph with degree sequence d into a graph satisfying the desired condition. Next we will show that every simple graph with degree sequence d can be transformed by such switches into every other.

1.3.32. Definition. A **2-switch** is the replacement of a pair of edges xy and zw in a simple graph by the edges yz and wx, given that yz and wx did not appear in the graph originally.



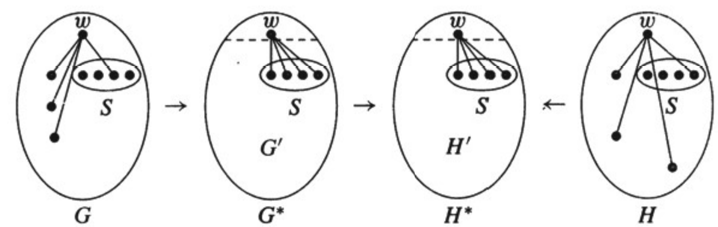
The dashed lines above indicate *nonadjacent* pairs. If $y \leftrightarrow z$ or $w \leftrightarrow x$, then the 2-switch cannot be performed, because the resulting graph would not be simple. A 2-switch preserves all vertex degrees. If some 2-switch turns H into H^*, then a 2-switch on the same four vertices turns H^* into H. Below we illustrate two successive 2-switches.



1.3.33.* Theorem. (Berge [1973, p153–154]) If G and H are two simple graphs with vertex set V, then $d_G(v) = d_H(v)$ for every $v \in V$ if and only if there is a sequence of 2-switches that transforms G into H.

Proof: Every 2-switch preserves vertex degrees, so the condition is sufficient. Conversely, when $d_G(v) = d_H(v)$ for all $v \in V$, we obtain an appropriate sequence of 2-switches by induction on the number of vertices, n. If $n \leq 3$, then for each $d_1, \dots, d_n$ there is at most one simple graph with $d(v_i) = d_i$. Hence we can use $n = 3$ as the basis step.

Consider $n \geq 4$, and let w be a vertex of maximum degree, Δ. Let $S = \{v_1, \dots, v_\Delta\}$ be a fixed set of vertices with the Δ highest degrees other than w. As in the proof of Theorem 1.3.31, some sequence of 2-switches transforms G to a graph G^* such that $N_{G^*}(w) = S$, and some such sequence transforms H to a graph H^* such that $N_{H^*}(w) = S$.



Since $N_{G^*}(w) = N_{H^*}(w)$, deleting w leaves simple graphs $G' = G^* - w$ and $H' = H^* - w$ with $d_{G'}(v) = d_{H'}(v)$ for every vertex v. By the induction hypothesis, some sequence of 2-switches transforms G' to H'. Since these do not involve w, and w has the same neighbors in G^* and H^*, applying this sequence transforms G^* to H^*. Hence we can transform G to H by transforming G to G^*, then G^* to H^*, then (in reverse order) the transformation of H to H^*. ■

We could also phrase this using induction on the number of edges appearing in exactly one of G and H, which is 0 if and only if they are already the same. In this approach, it suffices to find a 2-switch in G that makes it closer to H or a 2-switch in H that makes it closer to G.

EXERCISES

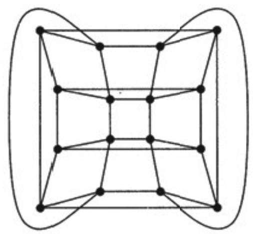
A statement with a parameter must be proved for all values of the parameter; it cannot be proved by giving examples. Counting a set includes providing proof.

1.3.1. (−) Prove or disprove: If u and v are the only vertices of odd degree in a graph G, then G contains a u,v-path.

1.3.2. (−) In a class with nine students, each student sends valentine cards to three others. Determine whether it is possible that each student receives cards from the same three students to whom he or she sent cards.

1.3.3. (−) Let u and v be adjacent vertices in a simple graph G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

1.3.4. (−) Prove that the graph below is isomorphic to Q_4.



1.3.5. (−) Count the copies of P_3 and C_4 in Q_k.

1.3.6. (−) Given graphs G and H, determine the number of components and maximum degree of $G + H$ in terms of the those parameters for G and H.

1.3.7. (−) Determine the maximum number of edges in a bipartite subgraph of P_n, of C_n, and of K_n.

1.3.8. (−) Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.

a) (5,5,4,3,2,2,2,1), c) (5,5,5,3,2,2,1,1),
b) (5,5,4,4,2,2,1,1), d) (5,5,5,4,2,1,1,1).

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1.3.9. In a league with two divisions of 13 teams each, determine whether it is possible to schedule a season with each team playing nine games against teams within its division and four games against teams in the other division.

1.3.10. Let l, m, n be nonnegative integers with $l + m = n$. Find necessary and sufficient conditions on l, m, n such that there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree.

1.3.11. Let W be a closed walk in a graph G. Let H be the subgraph of G consisting of edges used an odd number of times in W. Prove that $d_H(v)$ is even for every $v \in V(G)$.

1.3.12. (!) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $2k+1$-regular simple graph having a cut-edge.

1.3.13. (+) A **mountain range** is a polygonal curve from $(a, 0)$ to $(b, 0)$ in the upper half-plane. Hikers A and B begin at $(a, 0)$ and $(b, 0)$, respectively. Prove that A and B can meet by traveling on the mountain range in such a way that at all times their heights above the horizontal axis are the same. (Hint: Define a graph to model the movements, and use Corollary 1.3.5.) (Communicated by D.G. Hoffman.)

1.3.14. Prove that every simple graph with at least two vertices has two vertices of equal degree. Is the conclusion true for loopless graphs?

1.3.15. For each $k \geq 3$, determine the smallest n such that
a) there is a simple k-regular graph with n vertices.
b) there exist nonisomorphic simple k-regular graphs with n vertices.

1.3.16. (+) For $k \geq 2$ and $g \geq 2$, prove that there exists an k-regular graph with girth g. (Hint: To construct such a graph inductively, make use of an $k-1$-regular graph H with girth g and a graph with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. Comment: Such a graph with minimum order is a (k, g)**-cage**.) (Erdős–Sachs [1963])

1.3.17. (!) Let G be a graph with at least two vertices. Prove or disprove:
a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.
b) Deleting a vertex of degree $\delta(G)$ cannot reduce the average degree.

1.3.18. (!) For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

1.3.19. Let G be a claw-free graph. Prove that if $\Delta(G) \geq 5$, then G has a 4-cycle. For all $n \in \mathbb{N}$, construct a 4-regular claw-free graph of order at least n that has no 4-cycle.

1.3.20. (!) Count the cycles of length n in K_n and the cycles of length $2n$ in $K_{n,n}$.

1.3.21. Count the 6-cycles in $K_{m,n}$.

1.3.22. (!) Let G be a nonbipartite graph with n vertices and minimum degree k. Let l be the minimum length of an odd cycle in G.
a) Let C be a cycle of length l in G. Prove that every vertex not in $V(C)$ has at most two neighbors in $V(C)$.
b) By counting the edges joining $V(C)$ and $G - V(C)$ in two ways, prove that $n \geq kl/2$ (and thus $l \leq 2n/k$). (Campbell–Staton [1991])
c) When k is even, prove that the inequality of part (b) is best possible. (Hint: form a graph having $k/2$ pairwise disjoint l-cycles.)

1.3.23. Use the recursive description of Q_k (Example 1.3.8) to prove that $e(Q_k) = k2^{k-1}$.

1.3.24. Prove that $K_{2,3}$ is not contained in any hypercube Q_k.

1.3.25. (!) Prove that every cycle of length $2r$ in a hypercube is contained in a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r?

1.3.26. (!) Count the 6-cycles in Q_3. Prove that every 6-cycle in Q_k lies in exactly one 3-dimensional subcube. Use this to count the 6-cycles in Q_k for $k \geq 3$.

1.3.27. Given $k \in \mathbb{N}$, let G be the subgraph of Q_{2k+1} induced by the vertices in which the number of ones and zeros differs by 1. Prove that G is regular, and compute $n(G)$, $e(G)$, and the girth of G.

1.3.28. Let V be the set of binary k-tuples. Define a simple graph Q'_k with vertex set V by putting $u \leftrightarrow v$ if and only if u and v *agree* in exactly one coordinate. Prove that Q'_k is isomorphic to the hypercube Q_k if and only if k is even. (D.G. Hoffman)

1.3.29. (∗+) *Automorphisms of the k-dimensional cube* Q_k.
a) Prove that every copy of Q_j in Q_k is a subgraph induced by a set of 2^j vertices having specified values on a fixed set of $k - j$ coordinates. (Hint: Prove that a copy of Q_j must have two vertices differing in j coordinates.)
b) Use part (a) to count the automorphisms of Q_k.

1.3.30. Prove that every edge in the Petersen graph belongs to exactly four 5-cycles, and use this to show that the Petersen graph has exactly twelve 5-cycles. (Hint: For the first part, extend the edge to a copy of P_4 and apply Proposition 1.1.38.)

1.3.31. (!) Use complete graphs and counting arguments (not algebra!) to prove that
a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ for $0 \le k \le n$.
b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \le \binom{n}{2}$.

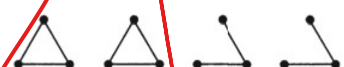
1.3.32. (!) Prove that the number of simple even graphs with vertex set $[n]$ is $2^{\binom{n-1}{2}}$. (Hint: Establish a bijection to the set of all simple graphs with vertex set $[n-1]$.)

1.3.33. (+) Let G be a triangle-free simple n-vertex graph such that every pair of non-adjacent vertices has exactly two common neighbors.
a) Prove that $n(G) = 1 + \binom{d(x)}{2}$, where $x \in V(G)$. Conclude that G is regular.
b) When $k = 5$, prove that deleting any one vertex and its neighbors from G leaves the Petersen graph. (Comment: When $k = 5$, the graph G is in fact the graph obtained from Q_4 by adding edges joining complementary vertices.)

1.3.34. (+) Let G be a kite-free simple n-vertex graph such that every pair of nonadjacent vertices has exactly two common neighbors. Prove that G is regular. (Galvin)

1.3.35. (+) Let n and k be integers such that $1 < k < n-1$. Let G be a simple n-vertex graph such that every k-vertex induced subgraph of G has m edges.
a) Let G' be an induced subgraph of G with l vertices, where $l > k$. Prove that $e(G') = m\binom{l}{k}/\binom{l-2}{k-2}$.
b) Use part (a) to prove that $G \in \{K_n, \overline{K}_n\}$. (Hint: Use part (a) to prove that the number of edges with endpoints u, v is independent of the choice of u and v.)

1.3.36. Let G be a 4-vertex graph whose list of subgraphs obtained by deleting one vertex appears below. Determine G.



1.3.37. Let H be a graph formed by deleting a vertex from a loopless regular graph G with $n(G) \ge 3$. Describe (and justify) a method for obtaining G from H.

1.3.38. Let G be a graph with at least 3 vertices. Prove that G is connected if and only if at least two of the subgraphs obtained by deleting one vertex of G are connected. (Hint: Use Proposition 1.2.29.)

1.3.39. (∗+) Prove that every disconnected graph G with at least three vertices is reconstructible. (Hint: Having used Exercise 1.3.38 to determine that G is disconnected, use $G_1, \dots, G_n$ to find a component M of G that occurs the most times among the components with the maximum number of vertices, use Proposition 1.2.29 to choose v so that $L = M - v$ is connected, and reconstruct G by finding some $G - v_i$ in which a copy of M became a copy of L.)

1.3.40. (!) Let G be an n-vertex simple graph, where $n \ge 2$. Determine the maximum possible number of edges in G under each of the following conditions.
a) G has an independent set of size a.
b) G has exactly k components.
c) G is disconnected.

1.3.41. (!) Prove or disprove: If G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

1.3.42. Let S be a set of vertices in a k-regular graph G such that no two vertices in S are adjacent or have a common neighbor. Use the pigeonhole principle to prove that $|S| \le \lfloor n(G)/(k+1) \rfloor$. Show that the bound is best possible for the cube Q_3. (Comment: The bound is not best possible for Q_4.)

1.3.43. (+) Let G be a simple graph with no isolated vertices, and let $a = 2e(G)/n(G)$ be the average degree in G. Let $t(v)$ denote the average of the degrees of the neighbors of v. Prove that $t(v) \ge a$ for some $v \in V(G)$. Construct an infinite family of connected graphs such that $t(v) > a$ for every vertex v. (Hint: For the first part, compute the average of $t(v)$, using that $x/y + y/x \ge 2$ when $x, y > 0$.) (Ajtai–Komlós–Szemerédi [1980])

1.3.44. (!) Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.
a) Prove that $G - x$ has average degree at least a if and only if $d(x) \le a/2$.
b) Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.
c) Show that there is no constant c greater than 1/2 such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible. (Hint: Use $K_{1,n-1}$.)

1.3.45. Determine the maximum number of edges in a bipartite subgraph of the Petersen graph.

1.3.46. Prove or disprove: Whenever the algorithm of Theorem 1.3.19 is applied to a bipartite graph, it finds the bipartite subgraph with the most edges (the full graph).

1.3.47. Use induction on $n(G)$ to prove that every nontrivial loopless graph G has a bipartite subgraph H such that H has *more* than $e(G)/2$ edges.

1.3.48. Construct graphs $G_1, G_2, \dots$ with G_n having $2n$ vertices, such that $\lim_{n\to\infty} f_n = 1/2$, where f_n is the fraction of $E(G_n)$ belonging to the largest bipartite subgraph of G_n.

1.3.49. For each $k \in \mathbb{N}$ and each loopless graph G, prove that G has a k-partite subgraph H (Definition 1.1.12) such that $e(H) \ge (1 - 1/k)e(G)$.

1.3.50. (+) For $n \ge 3$, determine the minimum number of edges in a connected n-vertex graph in which every edge belongs to a triangle. (Erdős [1988])

1.3.51. (+) Let G be a simple n-vertex graph, where $n > 3$.
a) Use Proposition 1.3.11 to prove that if G has more than $n^2/4$ edges, then G has a vertex whose deletion leaves a graph with more than $(n-1)^2/4$ edges. (Hint: In every graph, the number of edges is an integer.)
b) Use part (a) to prove by induction that G contains a triangle if $e(G) > n^2/4$.

1.3.52. Prove that every n-vertex triangle-free simple graph with the maximum number of edges is isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. (Hint: Strengthen the proof of Theorem 1.3.23.)

1.3.53. (!) Each game of *bridge* involves two teams of two partners each. Consider a club in which four players cannot play a game if two of them have previously been partners that night. Suppose that 15 members arrive, but one decides to study graph theory. The other 14 people play until each has been a partner with four others. Next they succeed in playing six more games (12 partnerships), but after that they cannot find four players containing no pair of previous partners. Prove that if they can convince the graph theorist to play, then at least one more game can be played. (Adapted from Bondy–Murty [1976, p111]).

1.3.54. (+) Let G be a simple graph with n vertices. Let $t(G)$ be the total number of triangles in G and $\overline{G}$ together.

a) Prove that $t(G) = \binom{n}{3} - (n-2)e(G) + \sum_{v \in V(G)} \binom{d(v)}{2}$ triangles. (Hint: Consider the contribution made to each side by each triple of vertices.

b) Prove that $t(G) \ge n(n-1)(n-5)/24$. (Hint: Use a lower bound on $\sum_{v \in V(G)} \binom{d(v)}{2}$ in terms of average degree.)

c) When $n-1$ is divisible by 4, construct a graph achieving equality in part (b). (Goodman [1959])

1.3.55. (+) *Maximum size with no induced P_4.*

a) Let G be the complement of a disconnected simple graph. Prove that $e(G) \le \Delta(G)^2$, with equality only for $K_{\Delta(G),\Delta(G)}$.

b) Let G be a simple connected P_4-free graph with maximum degree k. Prove that $e(G) \le k^2$. (Seinsche [1974], Chung–West [1993])

1.3.56. Use induction (on n or on $\sum d_i$) to prove that if $d_1, \dots, d_n$ are nonnegative integers and $\sum d_i$ is even, then there is an n-vertex graph with vertex degrees $d_1, \dots, d_n$. (Comment: This requests an alternative proof of Proposition 1.3.28.)

1.3.57. (!) Let n be a positive integer. Let d be a list of n nonnegative integers with even sum whose largest entry is less than n and differs from the smallest entry by at most 1. Prove that d is graphic. (Hint: Use the Havel–Hakimi Theorem. Example: 443333 is such a list, as is 33333322.)

1.3.58. *Generalization of Havel–Hakimi Theorem.* Given a nonincreasing list d of nonnegative integers, let d' be obtained by deleting d_k and subtracting 1 from the k largest elements remaining in the list. Prove that d is graphic if and only if d' is graphic. (Hint: Mimic the proof of Theorem 1.3.31.) (Wang–Kleitman [1973])

1.3.59. Define $d = (d_1, \dots, d_{2k})$ by $d_{2i} = d_{2i-1} = i$ for $1 \le i \le k$. Prove that d is graphic. (Hint: Do not use the Havel–Hakimi Theorem.)

1.3.60. (+) Let d be a list of integers consisting of k copies of a and $n-k$ copies of b, with $a \ge b \ge 0$. Determine necessary and sufficient conditions for d to be graphic.

1.3.61. (!) Suppose that $G \cong \overline{G}$ and that $n(G) \equiv 1 \bmod 4$. Prove that G has at least one vertex of degree $(n(G)-1)/2$.

1.3.62. Suppose that n is congruent to 0 or 1 modulo 4. Construct an n-vertex simple graph G with $\frac{1}{2}\binom{n}{2}$ edges such that $\Delta(G) - \delta(G) \le 1$.

1.3.63. (!) Let $d_1, \dots, d_n$ be integers such that $d_1 \ge \cdots \ge d_n \ge 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \dots, d_n$ if and only if $\sum d_i$ is even and $d_1 \le d_2 + \cdots + d_n$. (Hakimi [1962])

1.3.64. (!) Let $d_1 \le \cdots \le d_n$ be the vertex degrees of a simple graph G. Prove that G is connected if $d_j \ge j$ when $j \le n - 1 - d_n$. (Hint: Consider a component that omits some vertex of maximum degree.)

1.3.65. (+) Let $a_1 < \cdots < a_k$ be distinct positive integers. Prove that there is a simple graph with $a_k + 1$ vertices whose set of distinct vertex degrees is $a_1, \dots, a_k$. (Hint: Use induction on k to construct such a graph.) (Kapoor–Polimeni–Wall [1977])

1.3.66. (∗) *Expansion of 3-regular graphs* (see Example 1.3.26). For $n = 4k$, where $k \ge 2$, construct a connected 3-regular simple graph with n vertices that has no cut-edge but cannot be obtained from a smaller 3-regular simple graph by expansion. (Hint:

The desired graph must have no edge to which the inverse "erasure" operation can be applied to obtain a smaller simple graph.)

1.3.67. (∗) *Construction of 3-regular simple graphs*

a) Prove that a 2-switch can be performed by performing a sequence of expansions and erasures; these operations are defined in Example 1.3.26. (Caution: Erasure is not allowed when it would produce multiple edges.)

b) Use part (a) to prove that every 3-regular simple graph can be obtained from K_4 by a sequence of expansions and erasures. (Batagelj [1984])

1.3.68. (∗) Let G and H be two simple bipartite graphs, each with bipartition X, Y. Prove that $d_G(v) = d_H(v)$ for all $v \in X \cup Y$ if and only if there is a sequence of 2-switches that transforms G into H without ever changing the bipartition (each 2-switch replaces two edges joining X and Y by two other edges joining X and Y).

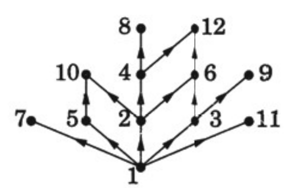
1.4. Directed Graphs

We have used graphs to model symmetric relations. Relation need not be symmetric; in general, a relation on S can be any set of ordered pairs in $S \times S$ (see Appendix A). For such relations, we need a more general model.

DEFINITIONS AND EXAMPLES

Seeking a graphical representation of the information in a general relation on S leads us to a model of directed graphs.

1.4.1. Example. For natural numbers x, y, we say that x is a "maximal divisor" of y if y/x is a prime number. For $S \subseteq \mathbb{N}$, the set $R = \{(x, y) \in S^2 \colon x \text{ is a maximal divisor of } y\}$ is a relation on S. To represent it graphically, we name a point in the plane for each element of S and draw an arrow from x to y whenever $(x, y) \in R$. Below we show the result when $S = [12]$. ■



1.4.2. Definition. A **directed graph** or **digraph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the **tail** of the edge, and the second is the **head**; together, they are the **endpoints**. We say that an edge is an edge **from** its tail **to** its head.

1.4.27. Definition. An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$. An **oriented graph** is an orientation of a simple graph. A **tournament** is an orientation of a complete graph.

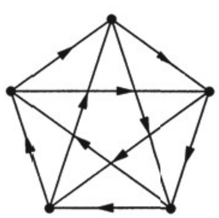
An oriented graph is the same thing as a loopless simple digraph. When the edges of a graph represent comparisons to be performed among items corresponding to the vertices, we can record the results by putting $x \to y$ when x does better than y in the comparison. The outcome is an orientation of G.

The number of oriented graphs with vertices $v_1, \ldots, v_n$ is $3^{\binom{n}{2}}$; the number of tournaments is $2^{\binom{n}{2}}$.

1.4.28. Example. Orientations of complete graphs model "round-robin tournaments". Consider an n-team league where each team plays every other exactly once. For each pair u, v, we include the edge uv if u wins or vu if v wins. At the end of the season we have an orientation of K_n. The "score" of a team is its outdegree, which equals its number of wins.

We therefore call the outdegree sequence of a tournament its **score sequence**. The outdegrees determine the indegrees, since $d^+(v) + d^-(v) = n - 1$ for every vertex v. It is easier to characterize the score sequences of tournaments than the degree sequences of simple graphs (Exercise 35). ■

A tournament may have more than one vertex with maximum outdegree, so there may be no clear "winner"—in the example below, every vertex has outdegree 2 and indegree 2. Choosing a champion when several teams have the maximum number of wins can be difficult. Although there need not be a clear winner, we show next that there must always be a team x such that, for every other team z, either x beats z or x beats some team that beats z.

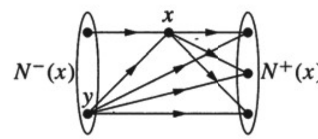


1.4.29. Definition. In a digraph, a **king** is a vertex from which every vertex is reachable by a path of length at most 2.

1.4.30. Proposition. (Landau [1953]) Every tournament has a king.

Proof: Let x be a vertex in a tournament T. If x is not a king, then some vertex y is not reachable from x by a path of length at most 2. Hence no successor of x is a predecessor of y. Since T is an orientation of a clique, every successor of x must therefore be a successor of y. Also $y \to x$. Hence $d^+(y) > d^+(x)$.

If y is not a king, then we repeat the argument to find z with yet larger outdegree. Since T is finite, we cannot forever obtain vertices of successively higher outdegree. The procedure must terminate, and it can terminate only when we have found a king. ■



In the language of extremality, we have proved that every vertex of maximum outdegree in a tournament is a king. Exercises 36–38 ask further questions about kings (see also Maurer [1980]). Exercise 39 generalizes the result to arbitrary digraphs.

EXERCISES

1.4.1. (−) Describe a relation in the real world whose digraph has no cycles. Describe another that has cycles but is not symmetric.

1.4.2. (−) In the lightswitch system of Application 1.4.4, suppose the first switch becomes disconnected from the wiring. Draw the digraph that models the resulting system.

1.4.3. (−) Prove that every u, v-walk in a digraph contains a u, v-path.

1.4.4. (−) Prove that every closed walk of odd length in a digraph contains the edges of an odd cycle. (Hint: Follow Lemma 1.2.15.)

1.4.5. (−) Let G be a digraph in which indegree equals outdegree at each vertex. Prove that G decomposes into cycles.

1.4.6. (−) Draw the de Bruijn graphs D_2 and D_4.

1.4.7. (−) Prove or disprove: If D is an orientation of a simple graph with 10 vertices, then the vertices of D cannot have distinct outdegrees.

1.4.8. (−) Prove that there is an n-vertex tournament with indegree equal to outdegree at every vertex if and only if n is odd.

• • • • •

1.4.9. For each $n \ge 1$, prove or disprove: Every simple digraph with n vertices has two vertices with the same outdegree or two vertices with the same indegree.

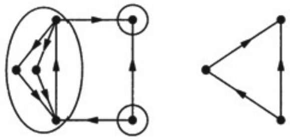
1.4.10. (!) Prove that a digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets S and T, there is an edge from S to T.

1.4.11. (!) Prove that in every digraph, some strong component has no entering edges, and some strong component has no exiting edges.

1.4.12. Prove that in a digraph the connection relation is an equivalence relation, and its equivalence classes are the vertex sets of the strong components.

1.4.13. a) Prove that the strong components of a digraph are pairwise disjoint.

b) Let $D_1, \dots, D_k$ be the strong components of a digraph D. Let D^* be the loopless digraph with vertices $v_1, \dots, v_k$ such that $v_i \to v_j$ if and only if $i \neq j$ and D has an edge from D_i to D_j. Prove that D^* has no cycle.



1.4.14. (!) Let G be an n-vertex digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, \dots, v_n$ so that if $v_iv_j \in E(G)$, then $i < j$.

1.4.15. Let G be the simple digraph with vertex set $\{(i, j) \in \mathbb{Z}^2 : 0 \le i \le m \text{ and } 0 \le n\}$ and an edge from (i, j) to (i', j') if and only if (i', j') is obtained from (i, j) by adding 1 to one coordinate. Prove that the number of paths from $(0, 0)$ to (m, n) in G is $\binom{m+n}{n}$.

1.4.16. (+) *Fermat's Little Theorem.* Let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo n (see Appendix A). Let a be a natural number having no common prime factors with n; multiplication by a defines a permutation of $\mathbb{Z}_n$. Let l be the least natural number such that $a^l \equiv a \bmod n$.

a) Let G be the functional digraph with vertex set $\mathbb{Z}_n$ for the permutation defined by multiplication by a. Prove that all cycles in G (except the loop on n) have length $l - 1$.

b) Conclude from part (a) that $a^{n-1} \equiv 1 \bmod n$.

1.4.17. (∗) Prove that a (directed) odd cycle is a digraph with no kernel. Construct a digraph that has an odd cycle as an induced subgraph but does have a kernel.

1.4.18. (∗) Prove that a digraph having no cycle has a unique kernel.

1.4.19. Use Lemma 1.4.23 and induction on the number of edges to prove the characterization of Eulerian digraphs (Theorem 1.4.24). (Hint: Follow Theorem 1.2.26.)

1.4.20. Prove the characterization of Eulerian digraphs (Theorem 1.4.24) using the notion of maximal trails. (Hint: Follow 1.2.32, the second proof of Theorem 1.2.26.)

1.4.21. Theorem 1.4.24 establishes necessary and sufficient conditions for a digraph to have an Eulerian circuit. Determine (with proof), the necessary and sufficient conditions for a digraph to have an Eulerian trail (Definition 1.4.22). (Good [1946])

1.4.22. Let D be a digraph with $d^-(v) = d^+(v)$ for every vertex v, except that $d^+(x) - d^-(x) = k = d^-(y) - d^+(y)$. Use the characterization of Eulerian digraphs to prove that D contains k pairwise edge-disjoint x, y-paths.

1.4.23. Prove that every graph G has an orientation D that is "balanced" at each vertex, meaning that $\left|d_D^+(v) - d_D^-(v)\right| \le 1$ for every $v \in V(G)$.

1.4.24. Prove or disprove: Every graph G has an orientation such that for every $S \subseteq V(G)$, the number of edges entering S and leaving S differ by at most 1.

1.4.25. (!) *Orientations and P_3-decomposition.*

a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Rotman [1991])

b) Use part (a) to conclude that a simple connected graph with an even number of edges can be decomposed into paths with two edges.

1.4.26. Arrange seven 0's and seven 1's cyclically so that the 14 strings of four consecutive bits are all the 4-digit binary strings other than 0101 and 1010.

1.4.27. *DeBruijn sequence for any alphabet and length.* Let A be an alphabet of size k. Prove that there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct. (Good [1946], Rees [1946])

1.4.28. Let S be an alphabet of size m. Explain how to produce a cyclic arrangement of $m^4 - m$ letters from S such that all four-letter strings of consecutive letters are different and contain at least two distinct letters.

1.4.29. (!) Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle. (Hint: Consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $(v_1, \dots, v_k)$ of G.)

1.4.30. (+) Given a strong digraph D, let $f(D)$ be the length of the shortest closed walk visiting every vertex. Prove that the maximum value of $f(D)$ over all strong digraphs with n vertices is $\lfloor (n+1)^2/4 \rfloor$ if $n \ge 2$. (Cull [1980])

1.4.31. Determine the minimum n such that there is a pair of nonisomorphic n-vertex tournaments with the same list of outdegrees.

1.4.32. Let $p = p_1, \dots, p_m$ and $q = q_1, \dots, q_n$ be lists of nonnegative integers. The pair (p, q) is **bigraphic** if there is a simple bipartite graph in which $p_1, \dots, p_m$ are the degrees for one partite set and $q_1, \dots, q_n$ are the degrees for the other. When p has positive sum, prove that (p, q) is bigraphic if and only if (p', q') is bigraphic, where (p', q') is obtained from (p, q) by deleting the largest element Δ from p and subtracting 1 from each of the Δ largest elements of q. (Hint: Follow the method of Theorem 1.3.31.)

1.4.33. (∗) Let A and B be two m by n matrices with entries in $\{0, 1\}$. An *exchange* operation substitutes a submatrix of the form $\binom{01}{10}$ for a submatrix of the form $\binom{10}{01}$ or vice versa. Prove that if A and B have the same list of row sums and have the same list of column sums, then A can be transformed into B by a sequence of exchange operations. Interpret this conclusion in the context of bipartite graphs. (Ryser [1957])

1.4.34. (!) Let G and H be two tournaments on a vertex set V. Prove that $d_G^+(v) = d_H^+(v)$ for all $v \in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3. (Hint: Consider a vertex of maximum outdegree in the subgraph of G consisting of edges oriented oppositely in H.) (Ryser [1964])

1.4.35. (+) Let $p_1, \dots, p_n$ be nonnegative integers with $p_1 \le \cdots \le p_n$. Let $p'_k = \sum_{i=1}^k p_i$. Prove that there exists a tournament with outdegrees $p_1, \dots, p_n$ if and only if $p'_k \ge \binom{k}{2}$ for $1 \le k < n$ and $p'_n = \binom{n}{2}$. (Hint: Use induction on $\sum_{k=1}^n [p'_k - \binom{k}{2}]$.) (Landau [1953])

1.4.36. By Proposition 1.4.30, every tournament has a king. Let T be a tournament having no vertex with indegree 0.

a) Prove that if x is a king in T, then T has another king in $N^-(x)$.

b) Use part (a) to prove that T has at least three kings.

c) For each $n \ge 3$, construct a tournament T with $\delta^-(T) > 0$ and only 3 kings.

(Comment: There exists an n-vertex tournament having exactly k kings whenever $n \ge k \ge 1$ except when $k = 2$ and when $n = k = 4$.) (Maurer [1980])

1.4.37. Consider the following algorithm whose input is a tournament T.

1) Select a vertex x in T.

2) If x has indegree 0, call x a king of T and stop.

3) Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T'.
4) Run the algorithm on T'; call the output a king in T and stop.
Prove that this algorithm terminates and produces a king in T

1.4.38. (+) For $n \in \mathbb{N}$, prove that there is an n-vertex tournament in which every vertex is a king if and only if $n \notin \{2, 4\}$.

1.4.39. (+) Prove that every loopless digraph D has a set S of pairwise nonadjacent vertices such that every vertex outside S is reached from S by a path of length at most 2. (Hint: Use strong induction on $n(D)$. Comment: This generalizes Proposition 1.4.30.) (Chvátal–Lovász [1974])

1.4.40. A directed graph is **unipathic** if for every pair of vertices x, y there is at most one (directed) x, y-path. Let T_n be the tournament on n vertices with the edge between v_i and v_j directed toward the vertex with larger index. What is the maximum number of edges in a unipathic subgraph of T_n? How many unipathic subgraphs are there with the maximum number of edges? (Hint: Show that the underlying graph has no triangles.) (Maurer–Rabinovitch–Trotter [1980])

1.4.41. Let G be a tournament. Let L_0 be a listing of $V(G)$ in some order. If y immediately follows x in L_0 but $y \to x$ in G, then yx is a **reverse edge**. We can interchange x and y in the order when yx is a reverse edge (this may increase the number of reverse edges). Suppose that a sequence $L_0, L_1, \cdots$ is produced by successively switching one reverse edge in the current order. Prove that this always leads to a list with no reverse edges. Determine the maximum number of steps to termination. (Comment: In the special case where the vertices are numbers and each edge points to the higher number of the pair, the result says that successively switching adjacent numbers that are out of order always eventually sorts the list.) (Locke [1995])

1.4.42. (!) Given an ordering $\sigma = v_1, \ldots, v_n$ of the vertices of a tournament, let $f(\sigma)$ be the sum of the lengths of the feedback edges, meaning the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Prove that every ordering minimizing $f(\sigma)$ places the vertices in nonincreasing order of outdegree. (Hint: Determine how $f(\sigma)$ changes when consecutive elements of σ are exchanged.) (Kano–Sakamoto [1983], Isaak–Tesman [1991])

Chapter 2

Trees and Distance

2.1. Basic Properties

The word "tree" suggests branching out from a root and never completing a cycle. Trees as graphs have many applications, especially in data storage, searching, and communication.

2.1.1. Definition. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph. A **leaf** (or **pendant vertex**) is a vertex of degree 1. A **spanning subgraph** of G is a subgraph with vertex set $V(G)$. A **spanning tree** is a spanning subgraph that is a tree.

2.1.2. Example. A tree is a connected forest, and every component of a forest is a tree. A graph with no cycles has no odd cycles; hence trees and forests are bipartite.

Paths are trees. A tree is a path if and only if its maximum degree is 2. A **star** is a tree consisting of one vertex adjacent to all the others. The n-vertex star is the biclique $K_{1,n-1}$.

A graph that is a tree has exactly one spanning tree; the full graph itself. A spanning subgraph of G need not be connected, and a connected subgraph of G need not be a spanning subgraph. For example:

If $n(G) > 1$, then the empty subgraph with vertex set $V(G)$ and edge set $\varnothing$ is spanning but not connected.

If $n(G) > 2$, then a subgraph consisting of one edge and its endpoints is connected but not spanning. ■